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Universität der Bundeswehr München

Physics based block preconditioning with sparse approximate inverses in MueLu

An application to beam solid interaction

Max Firmbach[1]

[1] Institute for Mathematics and Computer-Based Simulation (IMCS), University of the Bundeswehr Munich

Overview

1. Goal & Motivation
   Problem description

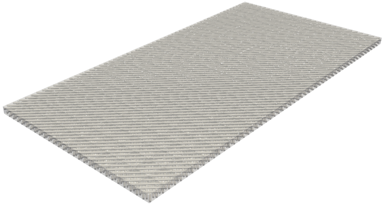
2. Schur-type block preconditioning in MueLu
   Basic multigrid algorithm   Building blocks in MueLu

3. Sparse Approximate Inverses
   Sparsity pattern selection

4. Application
   Weak scaling study   Composite plate   Reinforced concrete beam

Goal & Motivation

- Beam / Solid interactions occur in a wide variety of scenarios:
  - Engineering (steel-reinforced concrete, composite materials)
  - Biomechanics (collagen fibers in connective tissue)
- Time-to-solution dominated by cost for linear solver
  - Scalability through multilevel methods
  - Algebraic Multigrid (AMG) for its flexibility
  - But: Ill-conditioned matrix due to discretization and penalty regularization prohibit out-of-the-box block smoothing



Goal

Scalable AMG method for beam / solid interaction problems in penalty formulation

Problem description

The coupling of beam-like structures with solid continua is described with the following coupled linearized system for beam/solid interaction:

$$\begin{pmatrix} \mathbf{K}_S + \epsilon \mathbf{M}^T \kappa^{-1} \mathbf{M} & -\epsilon \mathbf{M}^T \kappa^{-1} \mathbf{D} \\ -\epsilon \mathbf{D}^T \kappa^{-1} \mathbf{M} & \mathbf{K}_B + \epsilon \mathbf{D}^T \kappa^{-1} \mathbf{D} \end{pmatrix} \begin{pmatrix} \Delta \mathbf{d}_S \\ \Delta \mathbf{d}_B \end{pmatrix} = - \begin{pmatrix} \mathbf{r}_S \\ \mathbf{r}_B \end{pmatrix}$$

Legend

$(.)_S$	solid contribution
$(.)_B$	beam contribution
$\mathbf{d}$	displacement DOFs
$\mathbf{r}$	residual
ϵ	penalty parameter
κ	scaling factor

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- **Solid DOFs**
- **Beam DOFs**
- **Coupling constraints**

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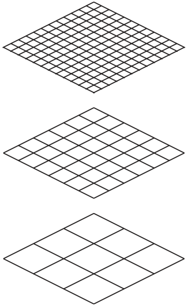
Challenges:

- Highly non-diagonal dominant and ill-conditioned block matrix due to penalty regularization
- Block matrix may be nonsymmetric due to beam formulation

Schur-type block preconditioning in MueLu

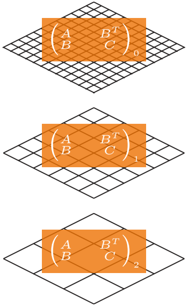
Basic multigrid algorithm



Basic ideas

- Attack different components of the error on different grids / levels.

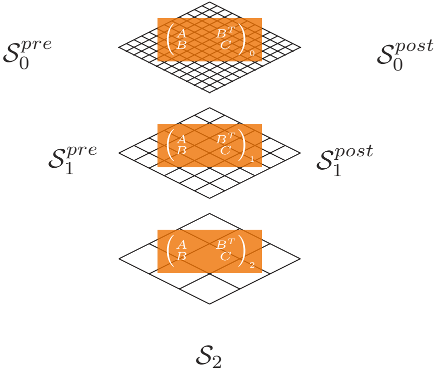
Basic multigrid algorithm



Basic ideas

- Attack different components of the error on different grids / levels.
- Reconstruct the fine level solution from information of coarse representations of the fine problem.

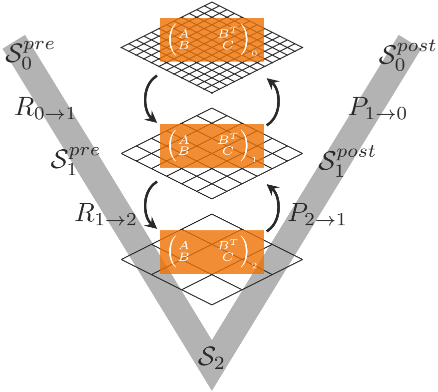
Basic multigrid algorithm



Basic ideas

- Attack different components of the error on different grids / levels.
- Reconstruct the fine level solution from information of coarse representations of the fine problem.
- Apply cheap **smoothers** on each multigrid level.

Basic multigrid algorithm



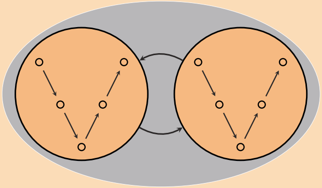
Basic ideas

- Attack different components of the error on different grids / levels.
- Reconstruct the fine level solution from information of coarse representations of the fine problem.
- Apply cheap **smoothers** on each multigrid level.
- **Restriction** and **prolongation operators** transfer information between different multigrid levels.

Approximations in Schur complement preconditioners

1. Approximation $\widehat{A} \approx A^{-1}$ to form Schur complement S
   - $\Rightarrow$ Governed by the type of block method
   - $\Rightarrow$ e.g. $\widehat{A} := diag(A)^{-1}$
2. Approximate block inverses within Schur complement preconditioner by standard AMG
   - $\Rightarrow$ Approximation quality can be controlled through the AMG settings

BlockMethod(AMG)[1]



- Coupling constraints are considered on fine level only
- Block method can be:
  - $\Rightarrow$ Block LU
  - $\Rightarrow$ Uzawa
  - $\Rightarrow$ SIMPLE

[1] Wiesner, T. A.; Mayr, M.; Popp, A.; Gee, M. W. and Wall, W. A. (2021): "Algebraic multigrid methods for saddle point systems arising from mortar contact formulations", *Numerical Methods in Engineering*, 122, 15:3749-3779

Buiding Blocks in MueLu

Factory Layout

BlockSmootherFactory
Block LU, Uzawa, SIMPLE

SmootherFactory2
Multigrid, ILU, ...

SubBlockFactory2

SmootherFactory1
Multigrid, ILU, ...

SchurComplementFactory

ApproximateInverseFactory
Diagonal, Lumping, Sparse Approximate Inverse

SubBlockFactory1

- Different Factories make up the Block-Smoother
- Can be specified with appropriate xml-file
- Several user-specific options

Sparse approximate invese

Due to the penalty regularization using just a diagonal approximation of the inverse inside the Schur complement calculation is not sufficient:

- Sparse approximate inverse methods (SPAI[2]) can produce better approximation
- Use matrix graph of A to calculate inverse $\widehat{A}$ on this sparsity pattern
- Based on Frobenius norm minimization:

$$\min_{\widehat{A} \in S} ||A\widehat{A} - I||_F$$

with S being the set of all sparse matrices with some known structure

Parallel computation

Decomposition into several least squares problems makes it inherently parallel:

$$||A\widehat{A} - I||_F^2 = \sum_{k=1}^{n} ||(A\widehat{A}_k - I)e_k||_2^2,$$

for each row k solve

$$\min_{\widehat{A}_k} ||A\widehat{A}_k - e_k||_2$$

with QR-decomposition

[2] Grothe, M. J. and Huckle, T. (1997): "Parallel preconditioning with sparse approximate inverses", *Journal Of Scientific Computing, 18, 3:838-853*

A priori pattern selection

Using just the pattern of A as input might not result in a satisfactory result, the matrix pattern needs to be enriched for a good sparse inverse approximation:

- Static approch by using recursive powers of graph of matrix $A \rightarrow$ recursion depth defined as level l
- Combining rows of graph $J(A)$ such that[3]:

$$J(A_k^l) = J(A_k^{l-1})J(A^{l-1})$$

- Pre- and post filtering of input graph and sparse inverse approximation with threshold value τ

SPAI with static pattern selection

1. Tresholding of $J(A)$
2. Determine graph of powers of A: $J(A^l)$
3. Calculate sparse inverse approximation $\widehat{A}$
4. Post filtering of $\widehat{A}$

[3]Chow E. (2001): "Parallel implementation and practical use of sparse approximate inverse preconditioners with a priori sparsity patterns", *The International Journal Of High Performance Computing Applications, 15:56-74*

Solid block with random fiber placement

Settings

Discretization

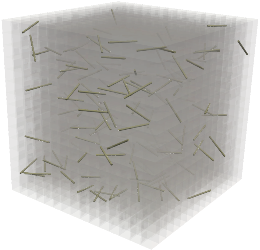
# Solid DOFs:	27783
# Beam DOFs:	1548
# procs:	1

Solver

Newton convergence:	10^{-6} (rel)
GMRES convergence:	10^{-8} (rel)

Material Parameters

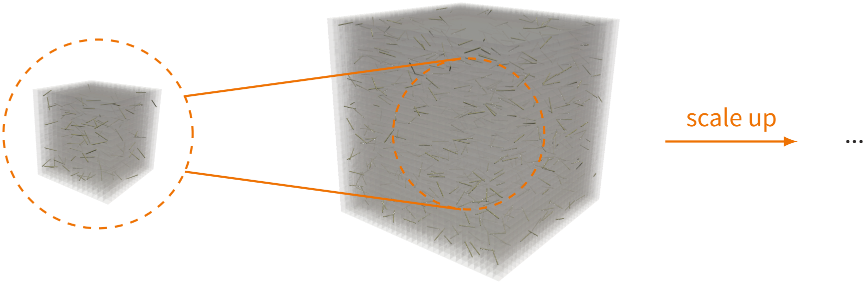
Solid:	$E_S = 1\frac{N}{m^2}, \nu_S = 0.3$ hyperelastic Saint Venant-Kirchhoff model
Beam:	$E_B = 10\frac{N}{m^2}, \nu_B = 0.0$ torsion-free Kirchhoff-Love model
Penalty:	$p = 10\frac{N}{m}$



- minimal working problem to be used for weak scaling study
- bottom surface is fixed, tensile surface load on top side

Weak scaling study

Weak scaling study: Cube filled with randomly placed and oriented fibers.

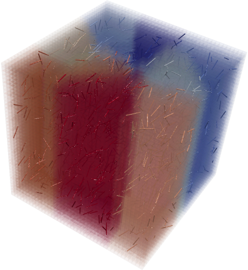


1 x 1 x 1 domain	2 x 2 x 2 subdomains	10 x 10 x 10 subdomains
• $\sim$ 25.000 Dofs • 1 Processor	• $\sim$ 200.000 Dofs • 8 Processors	• $\sim$ 25.000.000 Dofs • 1000 Processors

Weak scaling study

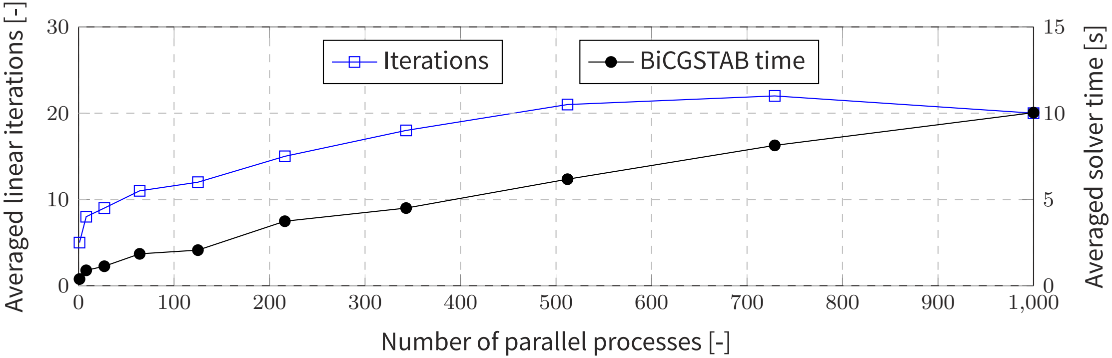
Weak scaling hierachy

ID	n^{proc}	n^{S}_{DOF}	n^{B}_{DOF}	n^{total}_{DOF}	$n^{total}_{DOF/proc}$
1	1	27783	1548	29331	29331.0
2	8	206763	14544	221307	27663.4
3	27	680943	52188	733131	27153.0
4	64	1594323	124788	1719111	26861.1
5	125	3090903	247560	3338463	26707.7
6	216	5314683	432300	5746983	26606.4
7	343	8409663	688956	9098619	26526.6
8	512	12519843	1035876	13555719	26476.1
9	729	17789223	1484736	19273959	26438.9
10	1000	24361803	2037192	26398995	26399.0



Domain decomposition approach based on a geometric bisection for $ID = 2$ with $n^{proc} = 8$

Weak scaling study



Fiber-Reinforced Composite Plate[4]

Settings

Discretization

Solid DOFs: 1950
Beam DOFs: 10992
procs: 6

Solver

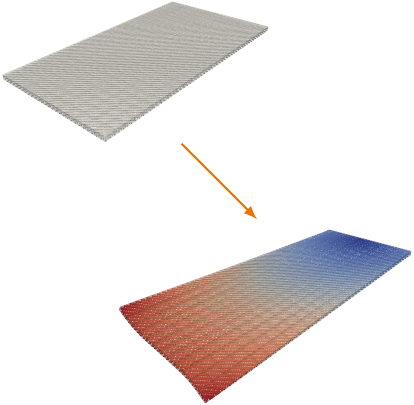
Newton convergence: 10^{-6} (rel)
BiCGSTAB convergence: 10^{-8} (rel)

Material Parameters

Solid: $E_S = 10\frac{N}{m^2}, \nu_S = 0.3$
hyperelastic Saint Venant-Kirchhoff model

Beam: $E_B = 1000\frac{N}{m^2}, \nu_B = 0.0$
torsion-free Kirchhoff-Love model

Penalty: $p = 1000\frac{N}{m}$



Deformation of the plate due to tensile load

[6] Steinbrecher, I.; Mayr, M.; Grill, M. J.; Kremheller, J.; Meier, C. and Popp, A. (2020): "A mortar-type finite element approach for embedding 1D beams into 3D solid volumes", *Computational Mechanics, 66:1377-1398*

Steel-Reinforced Concrete Beam

Settings

Discretization

# Solid DOFs:	5376
# Beam DOFs:	1686
# procs:	6

Solver

Newton convergence:	10^{-6} (rel)
BiCGSTAB convergence:	10^{-8} (rel)

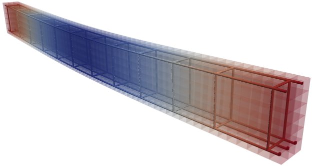
Material Parameters

Solid: $E_S = 30 \frac{N}{m^2}, \nu_S = 0.3$
hyperelastic Saint Venant-Kirchhoff model

Beam: $E_B = 210 \frac{N}{m^2}, \nu_B = 0.0$
torsion-free Kirchhoff-Love model

Penalty: $p = 1000 \frac{N}{m}$

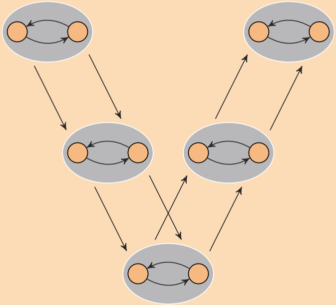
Four-point bending test under static loading[5]



[5] Braml, T.; Wimmer, J. and Varabei, Y. (2022): "Erfordernisse an die Datenaufnahme und -verarbeitung zur Erzeugung von intelligenten Digitalen Zwillingen", *Innsbrucker Bautage 2022 (eds Berger, J.) (Studia, 2022), 31-49*

Outlook

AMG(BlockMethod)



- Consider coupling constraints on all levels
- Assembly of the beam DOFs nullspace specific to beam formulation

- For now only considered torsion-free Kirchhoff–Love beam elements:
  - $\Rightarrow$ Sufficient for a broad range of applications
  - $\Rightarrow$ Restriction to straight center line in reference configuration

  Extend to other beam formulations in the near future.

Thank you!

Collaborators:

- Matthias Mayr
- Alexander Popp
- Ivo Steinbrecher

References:

Open-source implementation will be available in Trilinos/MueLu: https://trilinos.github.io/muelu.html

Contact:

- max.firmbach@unibw.de
- https://www.unibw.de/imcs-en